Comment on “Maximum likelihood reconstruction of CP maps”, quant-ph/0009104

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The treatment proposed by Sacchi quant-ph/0009104 does not represent correct solution, since the necessary conditions on CP maps are not guaranteed.

Maximum–likelihood (Max-Lik) principle finds wide variety of applications in quantum theory due to its ability to incorporate necessary conditions as constraints. Max-Lik has been devised for reconstruction of a generic quantum state keeping the positive semidefiniteness in Ref. [1]. Remarkably, it is not only an estimation, but a genuine generalized measurement [2]. In his recent papers Sacchi [3,4] applied Max-Lik to the reconstruction of CP maps using the numerical algorithm of downhill-simplex method [5]. However, the proposed treatment does not represent the correct Max-Lik solution consistent with quantum theory.

A trace preserving CP map is a linear map from operators in Hilbert space $\mathcal{H}$ to operators in $\mathcal{K}$. The mathematical formulation of CP map is expressed by the relations (3-5) of the paper [3]. The necessary conditions allowing physical interpretation are given by relations (6-8). This may be treated as a condition analogous to normalization of a density matrix $\mathrm{Tr}\varrho = 1$ in quantum state reconstruction. However, in the case of CP map such a condition is given by the relation (7) of [3]

$$S \geq 0, \quad \mathrm{Tr}_{\mathcal{K}} S = 1_{\mathcal{H}}. \tag{1}$$

The condition (1) effectively represents N^2 conditions, $N = \mathrm{dim}\mathcal{H}$. They all are correctly taken into account in Eq. (19) of Ref. [3] using the Lagrange multiplieres μ_{ij}

$$\mathcal{L}_{\mathrm{eff}}[S] = \mathcal{L}[S] - \mathrm{Tr}_{\mathcal{H}}[\mu(\mathrm{Tr}_{\mathcal{K}}[S])]. \tag{2}$$

Here $\mathcal{L}[S]$ denotes the log-likelihood given for example by the relation (2) of the paper [3]. Unfortunately, the problem of finding the maximum of $\mathcal{L}_{\mathrm{eff}}[S]$ under the constraints (1) has not been solved. Using an excuse that “multipliers cannot be easily inferred” the maximization has been done under “looser constraint” $\mathrm{Tr}S = N$ only. In this way, Sacchi has fixed the matrix of Lagrange multipliers as $\mu = (K/N)1_{\mathcal{H}}$, $K = \mathrm{dim}\mathcal{K}$. He was probably inspired by the numerical Max-Lik solution for quantum state estimation [5], where the semipositiveness and trace normalization represent the only constraints of quantum theory. However, this analogy is misleading and improper in the case of CP maps allowing also nonphysical solutions. Effectively, only a single condition instead of N^2 conditions is imposed in this case. As a consequence the reconstruction does not meet necessary conditions, namely the relation $\mathrm{Tr}_{\mathcal{K}}S = 1_{\mathcal{H}}$. This is obvious from the numerical results, which, in fact, may serve as a counterexample. The necessary conditions $S_p(1,1) + S_p(3,3) = 1$ and $S_p(2,2) + S_p(4,4) = 1$ are reproduced by the Table 1 as 0.995163231 and 1.006669754. In spite of the author’s claim that “the estimated values compare very well with the theoretical ones” the result does not correspond to any CP map. For example, the former relation means that the input state $|0\rangle$ is by such a “device” transformed into a state, where the total probability to appear on the output at the states $|0\rangle$ and $|1\rangle$ is only 0.995. The probability is therefore not conserved. Hence the condition (1) is as important as the semipositiveness itself.

Frankly, the proposed method hardly exhibits any significant advantage in comparison to recently used linear reconstruction methods. Linear approach is feasible for any dimension and all the necessary conditions are also fulfilled “approximately.” Linear treatment corresponds to the maximization of the likelihood (2) in [3] without any constraint. “Max-Lik” reconstruction of Sacchi [3] imposes semipositiveness and single constraint. This can be hardly considered as a significant difference since N^2 constraints must be taken into account in full quantum treatment.

Max-Lik reconstruction of CP maps can be formulated correctly [6]. The analogy between quantum state and CP map reconstructions is established on more sophisticated level than assumed in Ref. [3]. As shown in Ref. [6] solution for multiplieres and CP map may be obtained using an iterative algorithm. In the case of single qubit, the solution may be found even using numerical downhill simplex method, provided that the effective number of 12 parameters is used. Remarkably, the extremal equation has the form of closure relation for probability valued operator measure. The Max-Lik reconstruction can be therefore interpreted as a genuine quantum measurement in the same sense as in the case of quantum state reconstruction [2].

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